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(Unclassified Title)

STUDY OF THE SECOND-ORDER TERM IN THE MOD J 16 PIGA

(Unclassified)

ABSTRACT

The coefficient of the PIGA second-order nonlinearity term is developed in terms of the misalignment between the PIG and PIGA axes and the misalignment of the PIG axes with respect to the principal axes of inertia. The results indicate that to further reduce the nonlinearity term, a_2, equalization of the moments of inertia about SRA and IA is not profitable. Emphasis should be placed on dynamically balancing each of the rotating components, with a view towards reducing center of gravity shifts which contribute to the misalignment angle between the PIG IA and the principal axis.

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STUDY OF THE SECOND-ORDER TERM IN THE MOD J 16 PIGA

1. INTRODUCTION

The intent of this study is to define the contributions to the second-order term in the Mod J 16 PIGA. Several reports, as listed in the references, have dealt with torques due to float-wheel misalignments, mass unbalances, and flotation, but in view of the rigid specifications imposed on the Mod J 16 PIGA, a definitive study of second-order effects was undertaken. Therefore, the complete second-order term of a nonideal PIGA operating under a constant acceleration will be defined.

2. DEFINITION OF THE SECOND-ORDER TERM

The acceleration indicated by a PIGA output expressed in a power series is:

$$a_{ind} = a_o + a_1 a_{in} + a_2 a_{in}^2 + a_3 a_{in}^3 + \cdots \quad (1)$$

where:

a_{ind} = indicated PIGA acceleration output

a_{in} = acceleration input

$a_o, a_1, a_2, a_3 \cdots$ = coefficients

The coefficient a_2 is known as the second-order term. The torque summation equation about the PIG output axis under steady-state conditions is:

$$H W_{IA} = m\ell a_{in} + M_I \quad (2)$$

where:

$H W_{IA}$ = gyroscopic torque about OA

H = angular momentum of gyro wheel

W_{IA} = angular velocity of PIG about IA

$m\ell\, a_{in}$ = pendulous torque about OA

m = pendulous mass

ℓ = pendulous arm

a_{in} = component of acceleration along PIGA IA

M_I = inaccuracy torque

In this study, consideration is given to the inaccuracy torques dependent on the square of the applied acceleration only. Therefore, since the angular velocity of the PIG about IA is proportional to the applied acceleration, let:

$$M_I = \lambda W_{IA}^2 \tag{3}$$

where:

λ = constant of proportionality

Substituting Eq. (3) into Eq. (2):

$$\lambda W_{IA}^2 - H W_{IA} + m\ell\, a_{in} = 0 \tag{4}$$

Solving the quadratic equation (4) for W_{IA}:

$$W_{IA} = \frac{H}{2\lambda} \pm \frac{\sqrt{H^2 - 4\lambda\, m\ell\, a_{in}}}{2\lambda} \tag{5}$$

The only solution that describes the physical rotation of the PIG about IA is:

$$W_{IA} = \frac{H}{2\lambda} - \frac{\sqrt{H^2 - 4\lambda\, m\ell\, a_{in}}}{2\lambda} \tag{6}$$

The other solution results in an alternating series expansion for W_{IA}. Factoring Eq. (6):

$$W_{IA} = \frac{H}{2\lambda}\left[1 - \sqrt{1 - \left(\frac{4m\ell\, a_{in}}{H}\right)\left(\frac{\lambda}{H}\right)}\right] \tag{7}$$

Using an expansion series:

$$\sqrt{1 - x} = 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} \cdots \tag{8}$$

and letting $x = (4m\ell\, a_{in}/H)(\lambda/H)$, Eq. (7) becomes:

$$W_{IA} = \left(\frac{m\ell}{H}\right) a_{in} + \left(\frac{m\ell}{H}\right)^2 \left(\frac{\lambda}{H}\right) a_{in}^2 + 2\left(\frac{m\ell}{H}\right)^3 \left(\frac{\lambda}{H}\right)^2 a_{in}^3 \cdots \tag{9}$$

Referring to Eq. (1), the coefficient of the second-order term as defined by Eq. (9) is:

$$a_2 = \left(\frac{m\ell}{H}\right)^2 \left(\frac{\lambda}{H}\right) \tag{10}$$

Since the parameters m, ℓ, and H are defined by design, the parameter λ is the only undefined parameter required to define the second-order term.

3. DEVELOPMENT OF INACCURACY TORQUES AS A RESULT OF PIG TO PIGA MISALIGNMENT

Consider the orthogonal set of axes $0x'y'z'$ shown in Fig. 1 where the $0x'$ axis is defined as the PIGA input axis and where the angular velocity of the float about the PIGA IA is W. The total torque about the point 0 as a result of the angular velocity about the PIGA IA is:

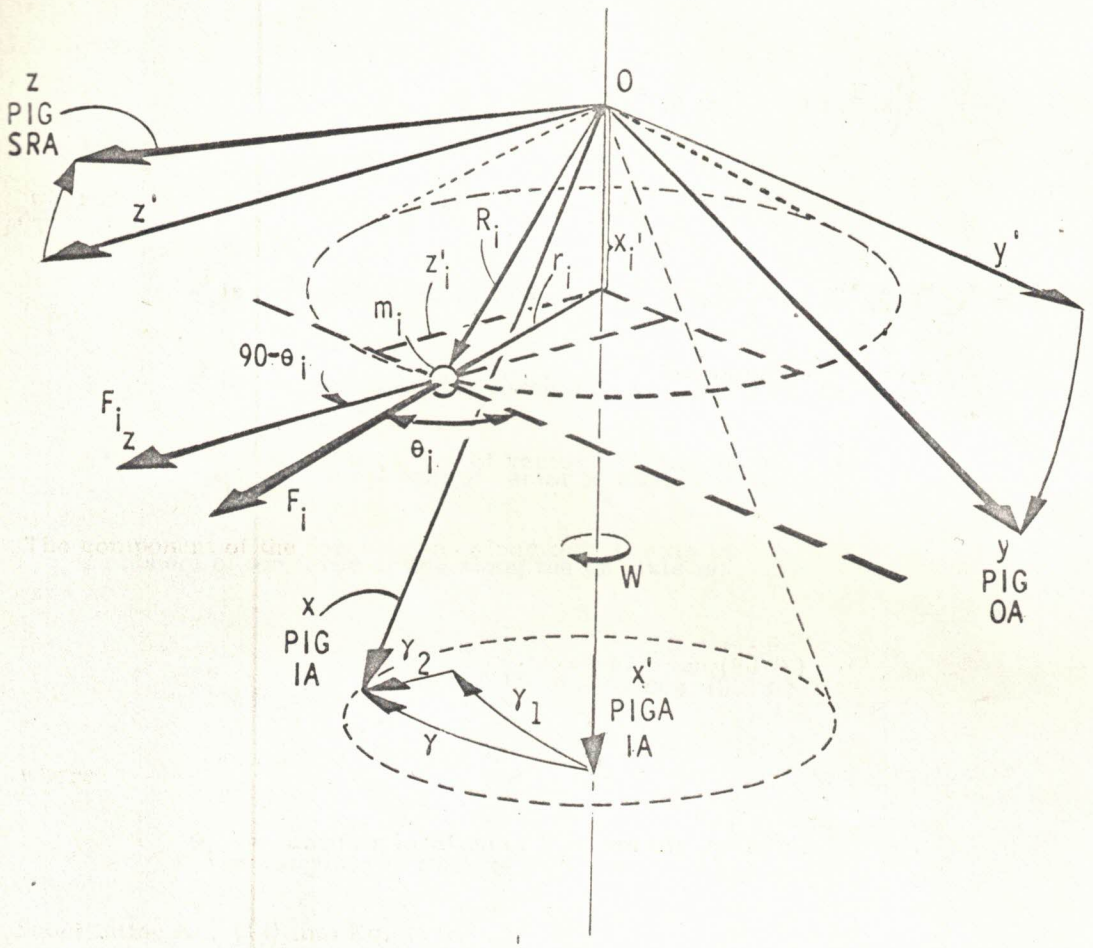
$$M_o = \sum_i R_i \times F_i \tag{11}$$

where:

M_o = torque about point 0

F_i = centrifugal force acting on the mass m_i

R_i = distance of mass m_i from point 0



γ = TOTAL MISALIGNMENT ANGLE BETWEEN PIG IA AND PIGA IA

γ_1 = COMPONENT OF PIG TO PIGA IA MISALIGNMENT ANGLE ABOUT SRA IN THE IA-OA PLANE

γ_2 = COMPONENT OF PIG TO PIGA IA MISALIGNMENT ANGLE ABOUT OA IN THE IA-SRA PLANE

Fig. 1 Components of the PIG to PIGA Misalignment Angle.

The centrifugal force, F_i, is:

$$F_i = m_i r_i W^2 \tag{12}$$

where r_i is the distance of the mass m_i from the $0x'$ axis and the summation is taken for all particles of mass m_i that make up the float structure. The component of the total torque about the $0y'$ axis is:

$$M_{y'} = \sum_i \left(z_i' F_{ix'} - x_i' F_{iz'} \right) \tag{13}$$

where:

$F_{ix'}$ = component of force F_i acting along the $0x'$ axis = 0

$F_{iz'}$ = component of force F_i acting along the $0z'$ axis

x_i' = component of vector R_i along the $0x'$ axis

The component of the force acting along the $0z'$ axis is:

$$F_{iz'} = F_i \cos (90-\theta_i) \tag{14}$$

where:

θ_i = angular location of F_i from the $0y'$ axis.

Substituting Eq. (12) into Eq. (14):

$$F_{iz'} = m_i r_i W^2 \sin \theta_i \tag{15}$$

Substituting Eq. (15) into Eq. (13) and, since $F_{ix'}$ equals zero, the torque about the $0y'$ axis becomes:

$$M_{y'} = -W^2 \sum_i x_i' m_i r_i \sin \theta_i \tag{16}$$

Since $z_i' = r_i \sin\theta_i$, Eq. (16) becomes:

$$M_y' = -W^2 \sum_i x_i' z_i' m_i \tag{17}$$

Therefore, the dynamic unbalance torque about the $0y'$ axis is:

$$M_y' = W^2 I_{x'z'} \tag{18}$$

where:

$I_{x'z'}$ = float product of inertia in the $x'z'$ plane = $-\sum m_i x_i' z_i'$

The torque about the float OA axis (0y axis) is:

$$M_{OA} = M_y = M_y' \cos A_{(\text{mis OA})} \tag{19}$$

where:

M_{OA} = torque about float OA, which is also defined as the 0y axis

A = misalignment angle between the 0y and $0y'$ axes (OA misalignment angle)

Neglecting the float OA misalignment angle:

$$M_{OA} = M_y = M_y' = W^2 I_{x'z'} \tag{20}$$

The float OA misalignment angle can be neglected in Eq. (19) because an OA misalignment angle that is $\leq$ 20 arc-sec, for example, results in an error of less than 0.05 ppm. To express the float product of inertia $I_{x'z'}$ in terms of the float coordinate frame IA, OA, SRA (x, y, z), through the PIG to PIGA misalignment angle γ (see Fig. 1), consider a matrix composed of two consecutive rotations:

(1) γ_1 = clockwise rotation about the $0z'$ axis

(2) γ_2 = clockwise rotation about the $0y'$ axis

The coordinates of m_i expressed in the transformed frame are:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\gamma_2 & 0 & \sin\gamma_2 \\ 0 & 1 & 0 \\ -\sin\gamma_2 & 0 & \cos\gamma_2 \end{bmatrix} \begin{bmatrix} \cos\gamma_1 & \sin\gamma_1 & 0 \\ -\sin\gamma_1 & \cos\gamma_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{21}$$

where x, y, z are the distances along the float IA, OA, and SRA axes, respectively.

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} x\cos\gamma_2\cos\gamma_1 + y\sin\gamma_1\cos\gamma_2 + z\sin\gamma_2 \\ -x\sin\gamma_1 + y\cos\gamma_1 \\ -x\sin\gamma_2\cos\gamma_1 - y\sin\gamma_1\sin\gamma_2 + z\cos\gamma_2 \end{bmatrix} \tag{22}$$

Using small angle approximations where the cos of an angle equals one and the sin of an angle equals the angle, Eq. (22) becomes:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} x + \gamma_1 y + \gamma_2 z \\ -\gamma_1 x + y \\ -\gamma_2 x - \gamma_1\gamma_2 y + z \end{bmatrix} \tag{23}$$

From Eq. (23), the product of $x'z'$ is:

$$x'z' = -\gamma_2 x^2 + \left(-2\gamma_1\gamma_2\right)xy + \left(1 - \gamma_2^2\right)xz + \left(-\gamma_1\gamma_2^2 + \gamma_1\right)yz + \gamma_2 z^2 - \gamma_1^2\gamma_2 y^2 \tag{24}$$

Neglecting products of angles, Eq. (24) may be expressed as:

$$x'z' = \gamma_2\left[\left(z^2 + y^2\right) - \left(x^2 + y^2\right)\right] + \gamma_1 yz + xz \tag{25}$$

The product of inertia in the $x'z'$ plane is:

$$I_{x'z'} = -\sum_i m_i x'_i z'_i \tag{26}$$

Substituting Eq. (25) into Eq. (26):

$$I_{x'z'} = \gamma_2 \sum m (x^2 + y^2) - \gamma_2 \sum m (z^2 + y^2) - \gamma_1 \sum myz - \sum mxz \tag{27}$$

$$I_{x'z'} = \gamma_2 (I_{zz} - I_{xx}) + \gamma_1 I_{yz} + I_{xz} \tag{28}$$

where:

I_{xx} = moment of inertia about the 0x axis

I_{zz} = moment of inertia about the 0z axis

I_{yz} = product of inertia in the yz plane

I_{xz} = product of inertia in the xz plane

Based on Eq. (20), the torque about the float output axis is:

$$M_{OA} = I_{x'z'} W^2 \tag{29}$$

Substituting Eq. (28) into Eq. (29), and neglecting $\gamma_1 I_{yz}$ with respect to the other terms, the portion of the error torque due to mass unbalance and due to the misalignment between the PIG and PIGA input axes is:

$$M_{OA} = \left[\gamma_2 \left(I_{zz} - I_{xx}\right) + I_{xz}\right] W^2 \tag{30}$$

4. DEVELOPMENT OF PIGA NONLINEARITY COEFFICIENT AS A RESULT OF PIG TO PIGA MISALIGNMENT AND ALIGNMENT TO FLOAT PRINCIPAL AXES

Consider the three coordinate systems shown in Fig. 2. The coordinate system $x'y'z'$ is an orthogonal set of axes with the $0x'$ axis defined as the PIGA IA. The coordinate system xyz is an orthogonal set of axes that defines the float IA, OA, and SRA, respectively. The coordinate system $x_p y_p z_p$ is an orthogonal set of axes that defines the principal axes of inertia of the float.

A rigid body rotating about a principal axis of inertia and not subject to an external torque

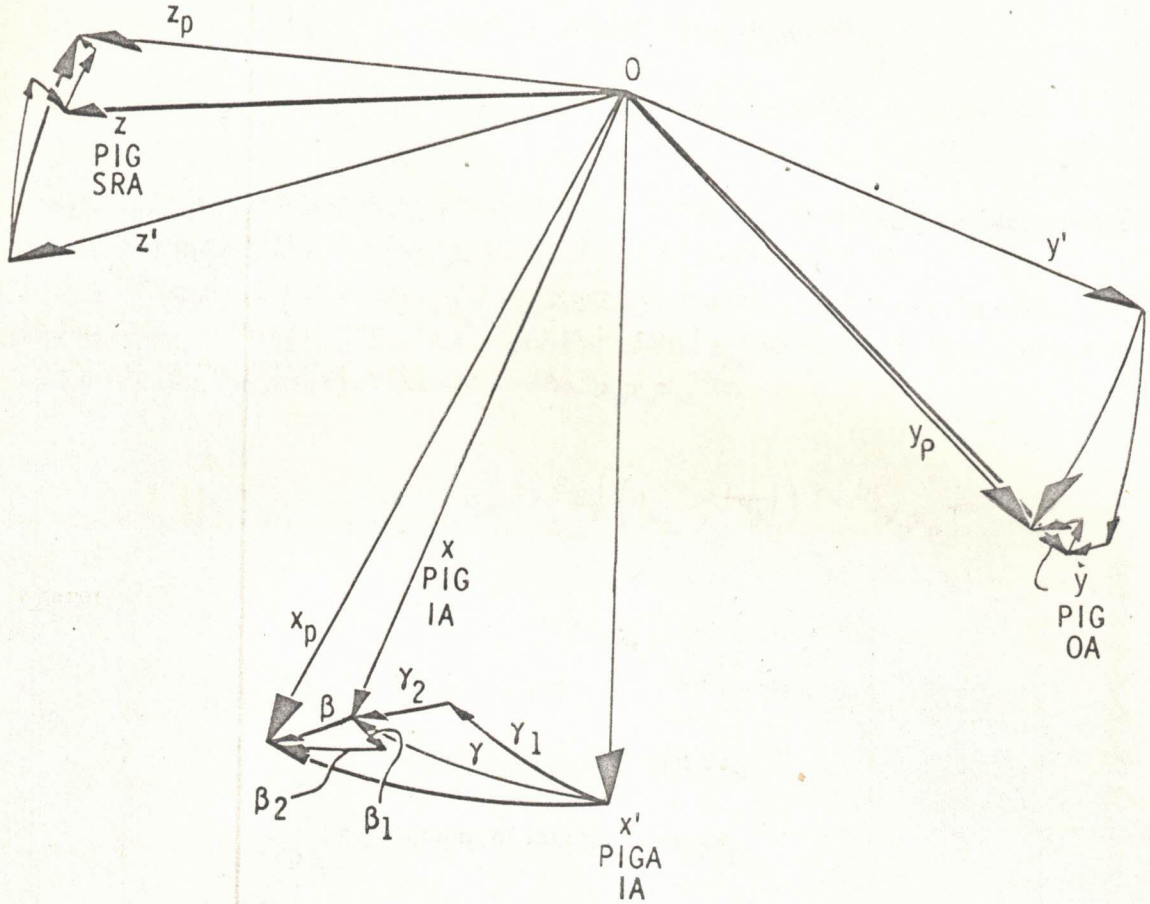


Fig. 2 Coordinate Systems Representing the PIGA Axes, the PIG Axes, and the Principal Axes of Inertia.

will rotate in equilibrium. In addition, stable equilibrium of a symmetrical body occurs only when rotation is about the principal axis of greatest moment of inertia. This is true of the 16 PIGA when:

$$I_{IA} > I_{SRA} > I_{OA} \tag{31}$$

where:

I_{IA} = moment of inertia about float IA

I_{SRA} = moment of inertia about float SRA

I_{OA} = moment of inertia about float OA

When the axis of rotation is not about a principal axis, a centrifugal couple, proportional to the square of the angular velocity, is exerted on the rotating member.

The product of inertia $I_{x'z'}$ expressed in terms of the xyz set of orthogonal axes is given in Eq. (28). In like manner, the product of inertia I_{xz} expressed in terms of the orthogonal axes representing the principal axes of inertia, $x_p y_p z_p$, is:

$$I_{xz} = \beta_2 \left(I_{z_p} - I_{x_p} \right) + I_{x_p z_p} \tag{32}$$

where:

β_2 = rotation angle about the 0y axis

I_{z_p} = moment of inertia about $0z_p$

I_{x_p} = moment of inertia about $0x_p$

$I_{x_p z_p}$ = product of inertia in the $x_p z_p$ plane

Since by definition of principal axes, all products of inertia vanish:

$$I_{x_p} \; I_{z_p} = 0 \tag{33}$$

Substituting Eq. (32) into Eq. (28), and assuming that $\left(I_z - I_x\right) = \left(I_{z_p} - I_{x_p}\right)$, Eq. (28) becomes:

$$I_{x'z'} = \left(I_{zz} - I_{xx}\right)\left(\gamma_2 + \beta_2\right) \tag{34}$$

Expressed in terms of the float axes, Eq. (34) is:

$$I_{x'z'} = \left(I_{SRA} - I_{IA}\right)\left(\gamma_2 + \beta_2\right) \tag{35}$$

where:

γ_2 = component of PIG IA to PIGA IA misalignment angle in the IA-SRA plane

β_2 = component of PIG IA to principal axis misalignment angle in the IA-SRA plane.

Substituting Eq. (35) into Eq. (29), the torque about the float or PIG OA is:

$$M_{OA} = \left[\left(I_{SRA} - I_{IA}\right)\left(\gamma_2 + \beta_2\right)\right] W_{IA}^2 \tag{36}$$

Substituting $M_{OA} = H\, W_{IA}$ and $W_{IA}^2 = (SF)^2\, a_{in}^2$ into Eq. (36):

$$H\, W_{IA} = (SF)^2 \left[\left(I_{SRA} - I_{IA}\right)\left(\gamma_2 + \beta_2\right)\right] a_{in}^2 \tag{37}$$

Referring to Eq. (10), the term λ is:

$$\lambda = \left(I_{SRA} - I_{IA}\right)\left(\gamma_2 + \beta_2\right) \tag{38}$$

and the second-order term is:

$$a_2 = (SF)^2 \frac{\lambda}{H}$$

$$a_2 = (SF)^2 \frac{\left(I_{SRA} - I_{IA}\right)\left(\gamma_2 + \beta_2\right)}{H} \frac{\text{rad/sec}}{\left(\text{cm/sec}^2\right)^2} \tag{39}$$

In terms of the more familar F_{i_1} coefficient:

$$F_{i_1} = (g)(SF)\frac{(I_{SRA} - I_{IA})(\gamma_2 + \beta_2)}{H} \times 10^6 \text{ ppm/g} \quad (40)$$

where:

$$g = 980.39215 \text{ cm/sec}^2$$

$$SF = 0.00102 \text{ (rad/sec)/(cm/sec}^2\text{)}$$

5. METHOD FOR ESTIMATING COMPONENT OF MISALIGNMENT ANGLE BETWEEN PIG IA AND PRINCIPAL AXIS OF INERTIA IN THE IA-SRA PLANE

In the expression of the second-order term, Eq. (39), the terms SF, I_{SRA}, I_{IA}, and H are defined by design, and γ_2, the component of the total input axis misalignment angle γ in the IA-SRA plane, is adjusted to a minimum during assembly (see reference 5). A method of estimating the remaining term in Eq. (39), β_2, is described herein. The sequence of steps required to estimate β_2 is as follows:

(a) Measure I_{OA}, I_{IA}, and I_{SRA} of the float.

(b) Estimate I_{SRA-OA}, I_{SRA-IA}, and I_{IA-OA} of the float.

(c) Form the float inertia matrix I_F.

(d) Subject the float inertia matrix to a principal axis transform.

(e) Obtain the direction cosines of the principal axes of inertia $x_p y_p z_p$ relative to the float axes xyz.

(f) Calculate $\beta_2 = 1/2\,\beta$.

(1) The moments of inertia about the float axes, I_{OA}, I_{IA}, and I_{SRA} can be measured to an accuracy of ± 0.04%, using a torsional pendulum (see reference 6).

(2) The products of inertia I_{SRA-OA}, I_{SRA-IA}, and I_{IA-OA} can be obtained only by estimating the minimum and maximum values. The values used were 0.001 and 0.1 gm-cm^2.

(3) The float inertia matrix is formed as:

$$I_F = \begin{bmatrix} I_{OA} & I_{OA-IA} & I_{OA-SRA} \\ I_{OA-IA} & I_{IA} & I_{SRA-IA} \\ I_{OA-SRA} & I_{SRA-IA} & I_{SRA} \end{bmatrix} \quad (41)$$

(Since I_F is composed of coefficients of inertia, I_F is symmetric.)

(4) The principal axis transform consists of diagonalizing I_F, giving the three moments of inertia about the principal axes. To diagonalize I_F, solve the cubic equation in I resulting from the determinant

$$\begin{vmatrix} I_{OA}-I & I_{OA-IA} & I_{OA-SRA} \\ I_{OA-IA} & I_{IA}-I & I_{SRA-IA} \\ I_{OA-SRA} & I_{SRA-IA} & I_{SRA}-I \end{vmatrix} = 0 \qquad (42)$$

The three roots are the three principal moments of inertia I_{p1}, I_{p2}, and I_{p3}.

(5) The direction cosines are obtained from the matrix transformation that transfers the xyz axes into the alignment with $x_p y_p z_p$ axes. This is obtained by solving the following simultaneous equations for the ratios y/x and z/x.

$$\left(I_{OA} - I_{p1}\right) x + I_{OA-IA}\, y + I_{OA-SRA}\, z = 0 \qquad (43)$$

$$I_{OA-IA}\, x + \left(I_{IA} - I_{p1}\right) y + I_{SRA-IA}\, z = 0 \qquad (44)$$

The result is normalized by setting $x = 1.0$ and the angle β is calculated from:

$$\cos\beta = \frac{x}{\sqrt{x^2 + y^2 + z^2}} \qquad (45)$$

(6) The angle β_2 is calculated as:

$$\beta_2 = 1/2\,\beta \qquad (46)$$

6. CONCLUSIONS

As indicated by Eq. (40), only the components of the misalignment angles γ and β in the SRA - IA plane contribute to F_{i_1}. In all the calculations, the assumptions were made that $\beta_2 = 1/2\,\beta$ and $\gamma_2 = 1/2\,\gamma$. A computer program was written for the sequence given in paragraph 5 to determine β_2 for various PIGA units with products of inertias varying from 0.001 to 0.1 gm-cm^2. The results of this program and the calculations of the F_{i_1} term are given in Table 1, Fig. 3, and Fig. 4.

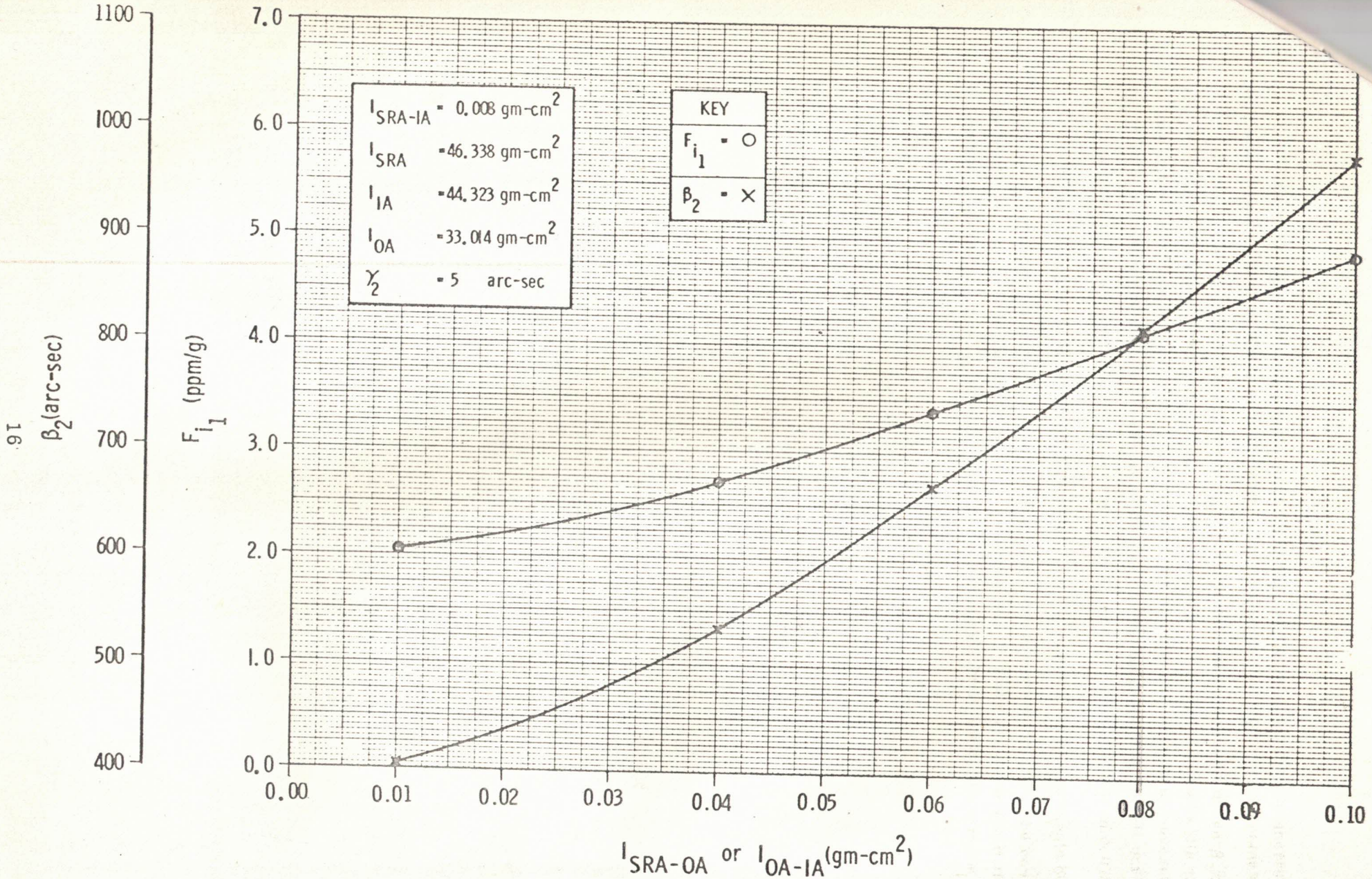


Fig. 4 Misalignment Angle β_2 and Nonlinearity Coefficient F_{i_1} as Functions of I_{SRA-OA} or I_{OA-IA} With I_{SRA-IA} = 0.008 gm-cm^2.

Equation (40) seems to indicate that mere equalization of the moments of inertia about SRA and IA is sufficient to control the F_{i_1} term. This assumes that β_2 remains constant as I_{SRA} and I_{IA} are changed to reduce their difference. This assumption is incorrect because β_2 increases rapidly as the difference between I_{SRA} and I_{IA} decreases (see Fig. 3). As shown in Fig. 3, a decrease in the difference of I_{SRA} and I_{IA} from 2.015 to 0.154 gm-cm^2 results in a decrease in the F_{i_1} term of an average of only 0.075 ppm/g. For $I_{OA=IA}$ and $I_{SRA=OA}$ less than 0.01 gm-cm^2 (which is probably more correct), the decrease in the F_{i_1} term is much smaller than 0.075 ppm/g.

The conclusion is reached that the main effort in reducing the F_{i_1} term should be directed toward dynamic balancing of each rotating component to reduce the shifts in the center of gravity that contribute to the angle β_2, (see reference 6). Further reduction in the inertia difference between I_{SRA} and I_{IA} is not profitable due to the relatively little effect on the F_{i_1} term.

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